The Lyapunov Exponent Relation in the Outer Asteroid Belt

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Outline

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- Phase Space Diffusion
- Lyapunov Exponent
- Lyapunov Exponent Relation
  - definition
  - previous results
  - current problems
- New Results
  - Severe Chaos
  - Overlap Threshold
  - Stable Regions
  - Mean-Motion Resonances

Outer Asteroid Belt

- Asteroid distribution

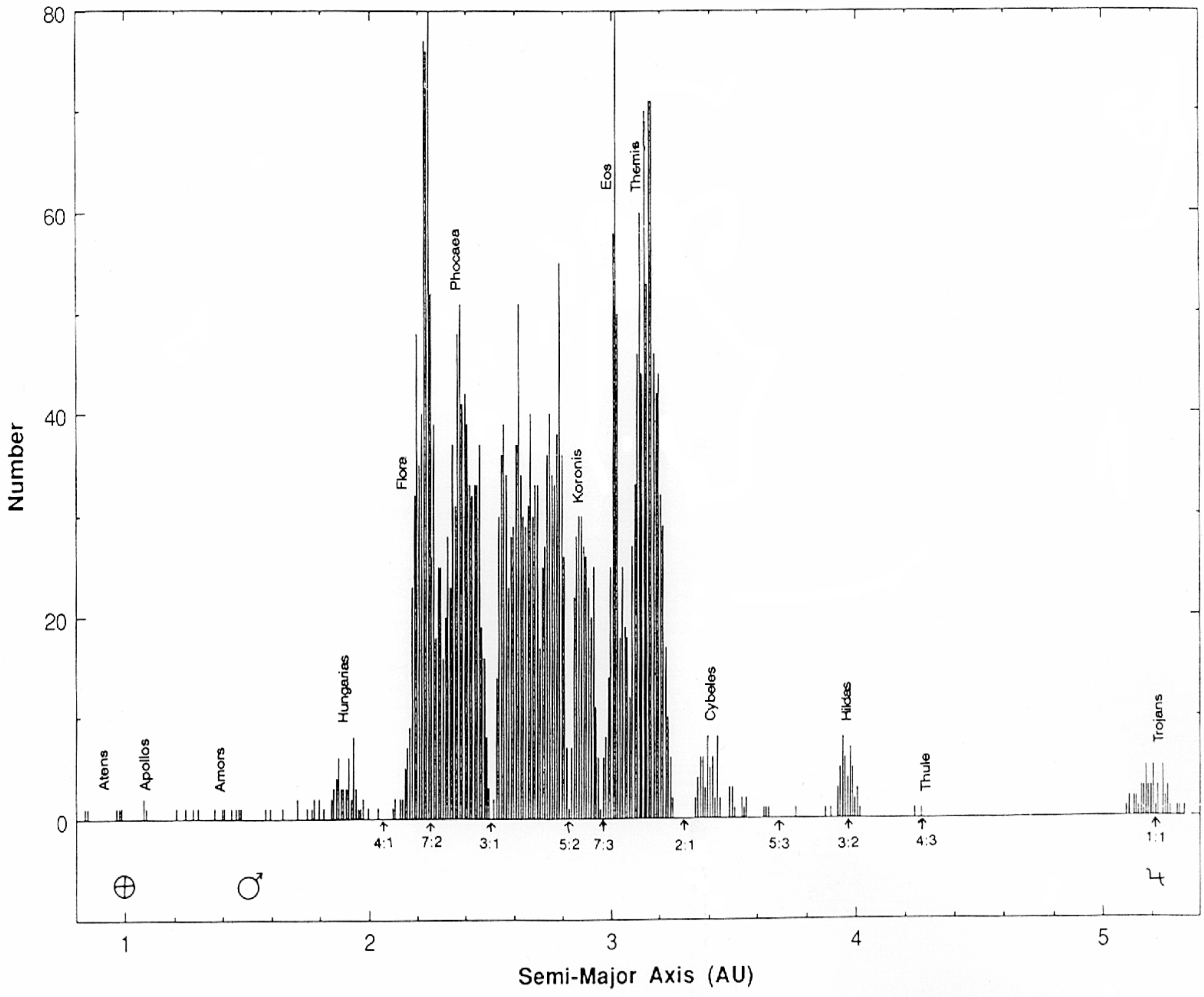
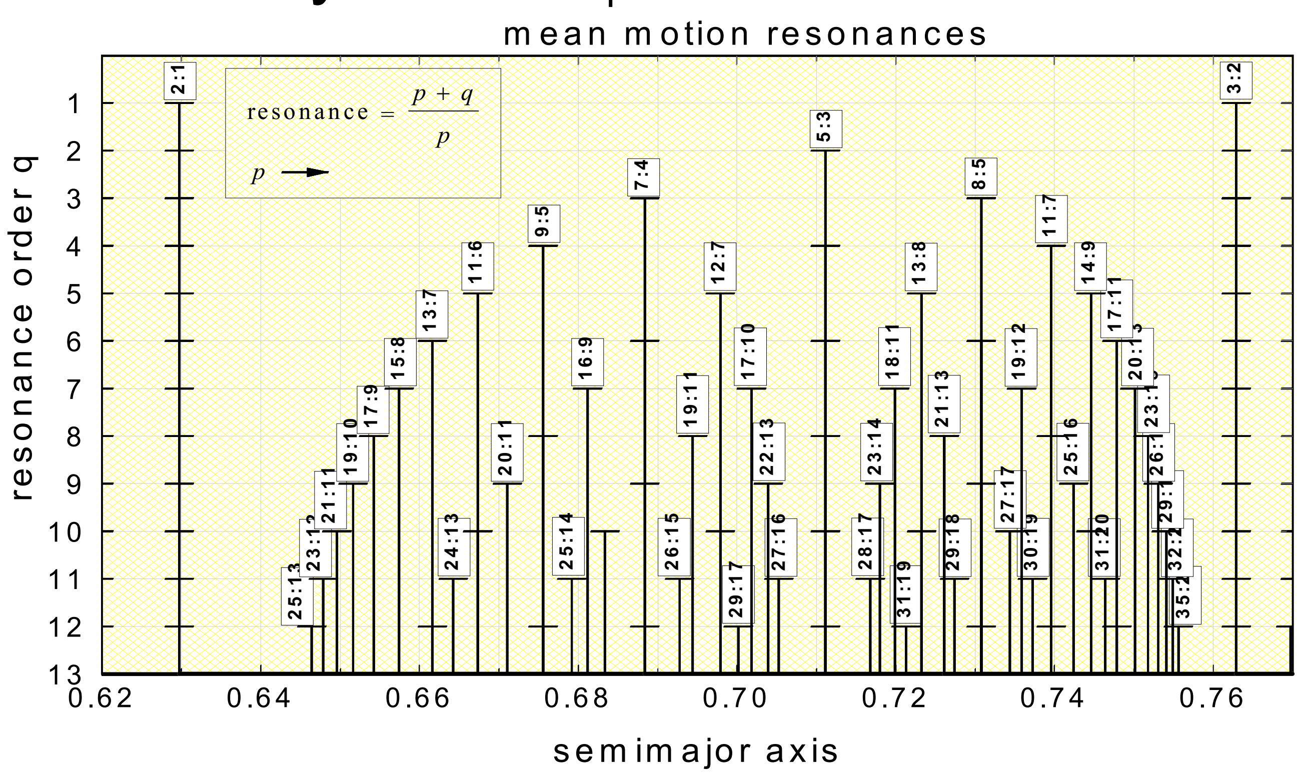


Fig. 2. Heliocentric distribution of orbital semimajor axes for nearly 4000 numbered asteroids. Commonly referred to regions and the major Jovian resonances are labeled. Other frequently referred to zones (such as those defined by Zellner et al. 1985*a*) and their eccentricity and inclination boundaries are listed in Table I of the chapter by Gradie et al. Compare this diagram with that on the back cover (which depicts the actual asteroid positions at a given moment in time) to see how orbital eccentricities tend to "smear" this distribution.

Outer Asteroid Belt

- $0.65 \lesssim a/a_{Jup} \lesssim 0.75$ (between 2:1 and 3:2)
- No major resonances
- Permeated with isolated high-order mean-motion resonances
  - As is inner solar system (see Whipple 1995)
- Lies inside the resonance overlap boundary at $a/a_{Jup} \approx 0.79$



Phase Space Diffusion

- 2 Degrees of Freedom
  - mathematics well in hand
  - *isolating* surfaces
  - turnstyles
  - cantori (fractal holes)
    - Stable islands surrounded by an infinite set of ever more porous fractal sieves
    - Those closest to islands are least porous
    - $P(t) \sim t^{-\beta}$ ($\beta \approx 1.4$)

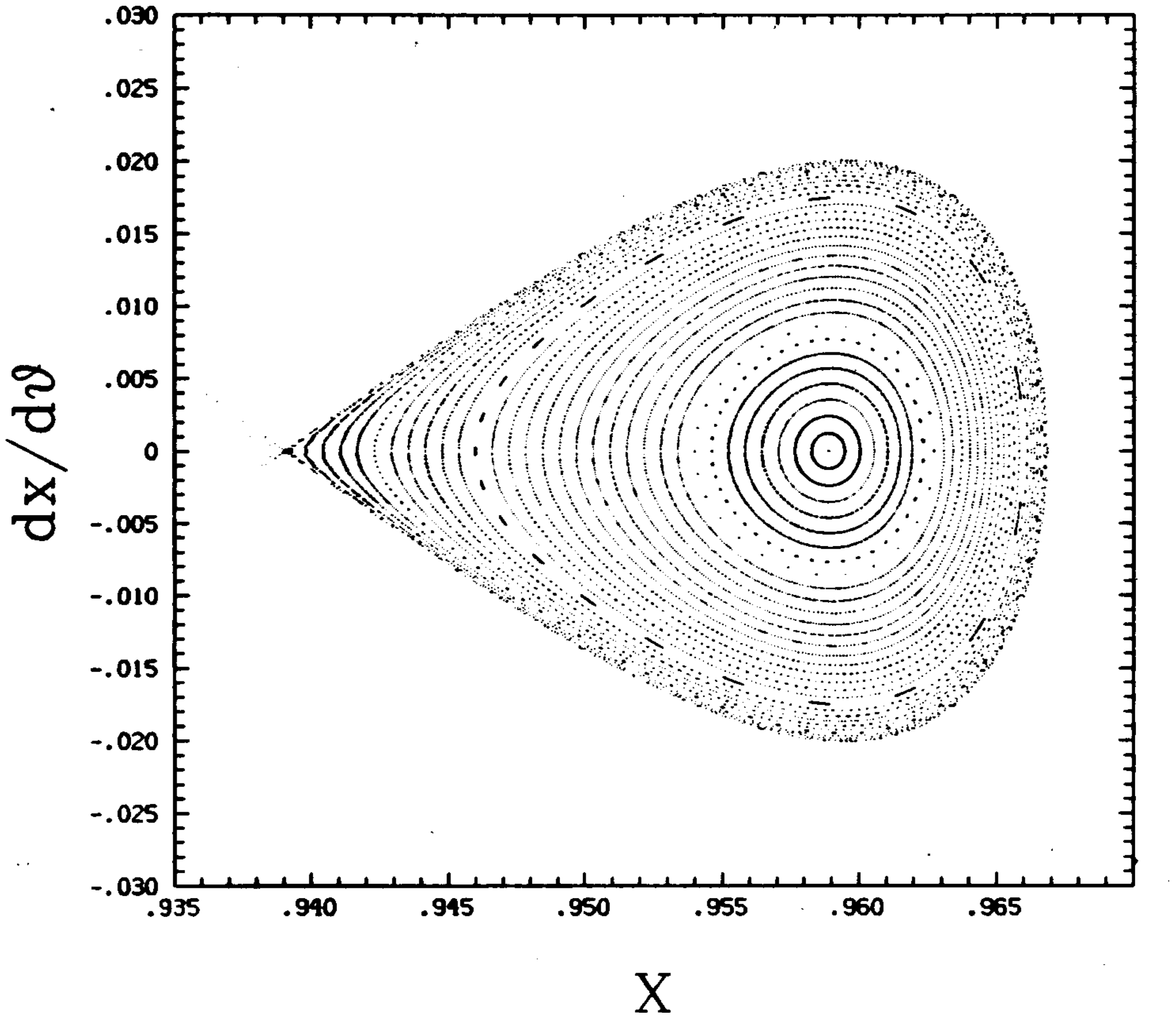
Phase Space Diffusion

- sticky tori
  - Orbits near an invariant surface can become trapped for long times
  - Higher resonance order = more "sticky"
  - Lower bound: $$T \geq \frac{C}{r} e^{\left(\frac{K\beta}{r}\right)^{\alpha}}$$
    - irrationality b > 0, a = 1/(N+2)
    - Nekhoroshev (1977)

- ≥3 Degrees of Freedom
  - mathematics is formidable
  - turnstyle analogs exist (Wiggins 1992)
  - normally hyperbolic invariant manifolds can play the role of separatrices

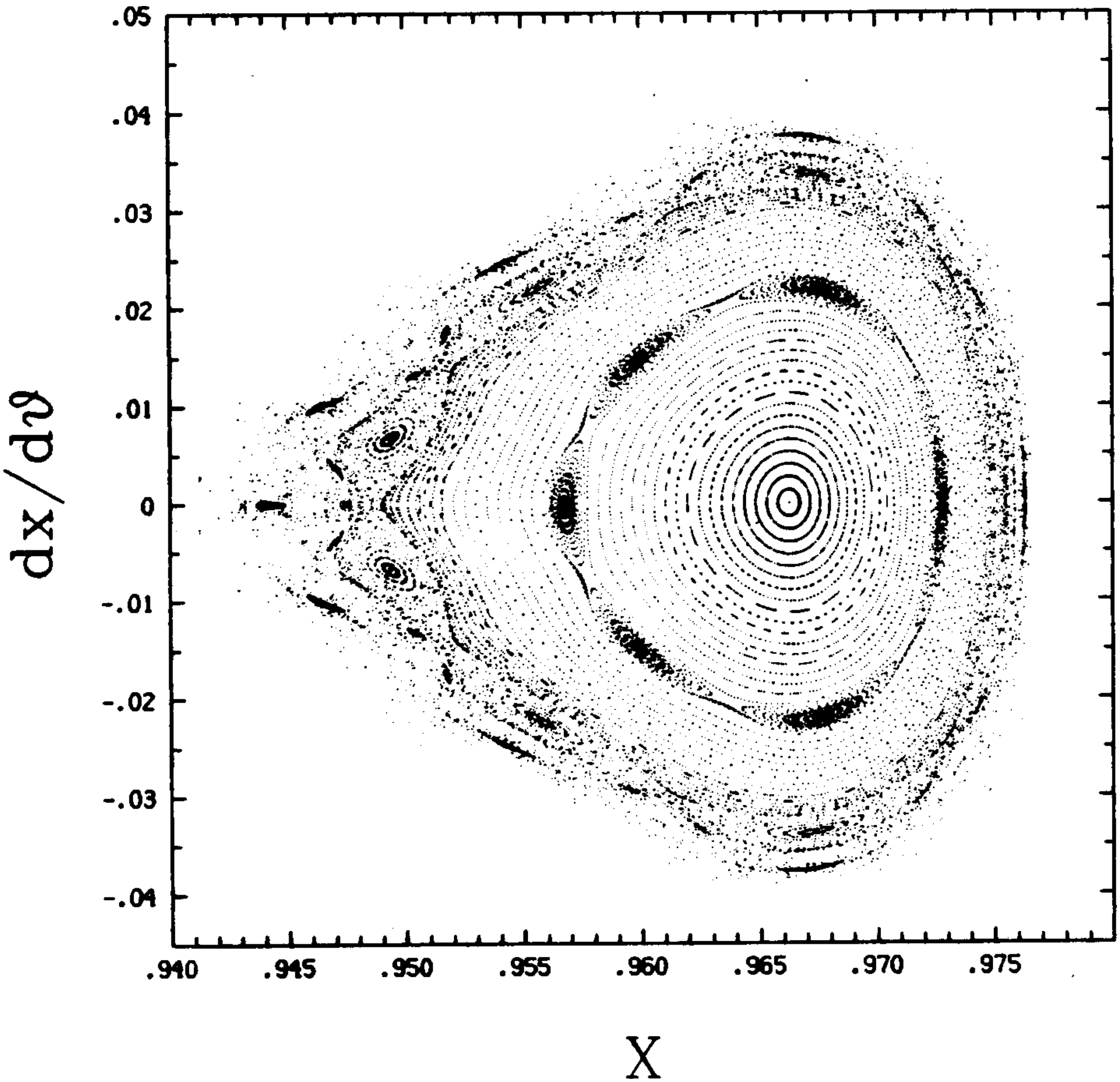
2 Dynamical Regimes

- Resonance Overlap Regime
  - elliptic and hyperbolic fixed points
  - separatrices



2 Dynamical Regimes

- (Resonance Overlap)
  - hypersurface destruction



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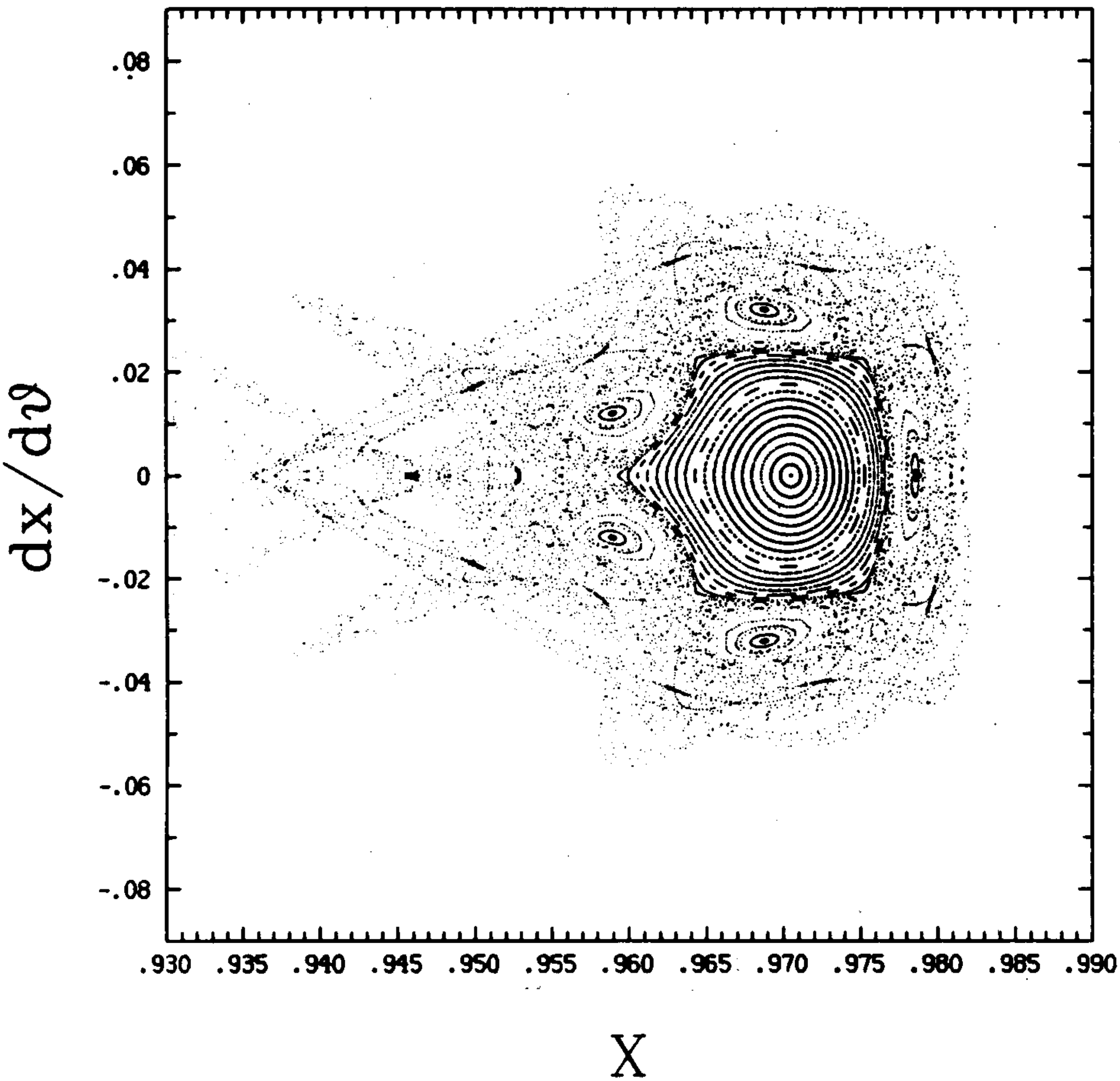

2 Dynamical Regimes

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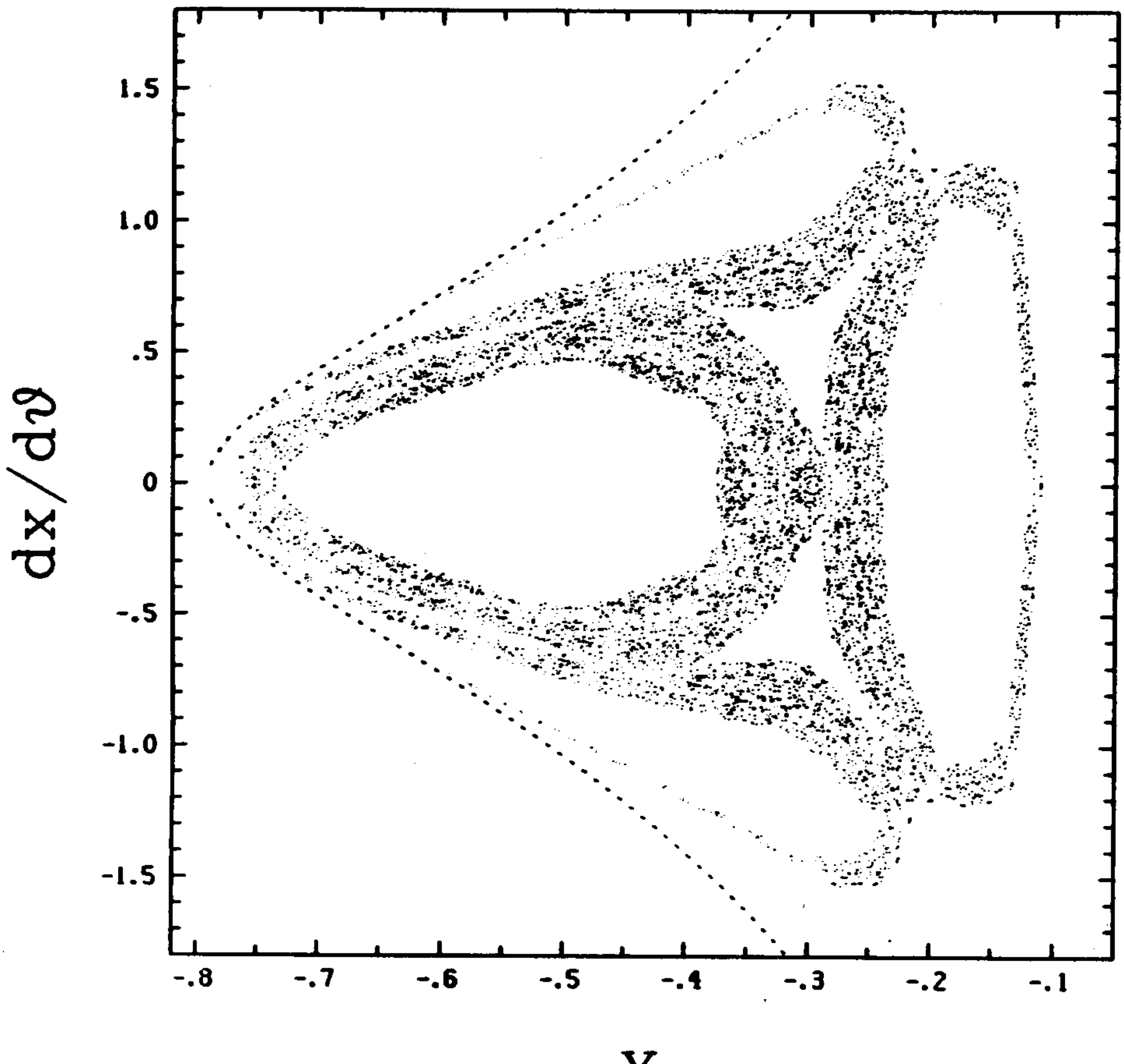


2 Dynamical Regimes

- (Resonance Overlap)
  - global chaos
    - Dynamics governed by a FP diffusion equation for the distribution function in action space:

$$\frac{\partial N(I,t)}{\partial t} = \frac{\partial}{\partial I}\left[D \frac{\partial N(I,t)}{\partial I} \right] - \frac{N(I,t)}{T_e}$$

    - Numerical experiments (Konishi 1989):

$$\log D \approx a + b \log \lambda$$

    - Varvoglis & Anastasiadis (1996):

$$\log T_e \approx a + b \log T_L$$

2 Dynamical Regimes

- Isolated (i.e., nonoverlapping) resonances
  - Lévy statistics
    - Used to describe fractal (scale invariant) random processes
    - Mean-squared jump distance:

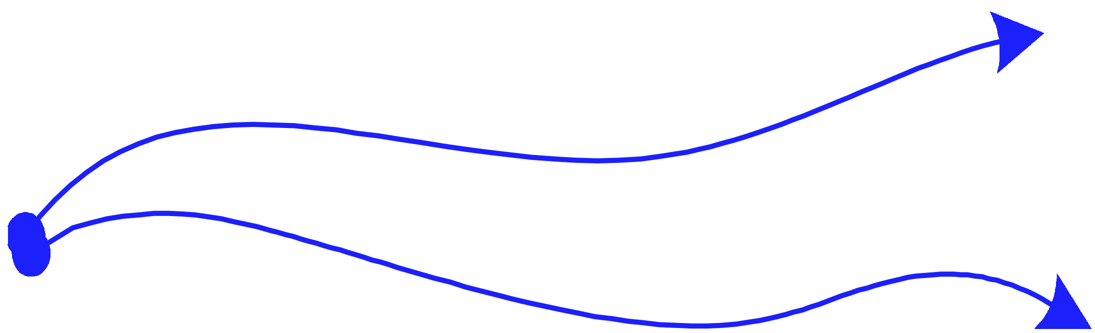
$$\left\langle r^2(t) \right\rangle \sim t^{\gamma}$$

      - $\gamma = 1$ Brownian motion
      - $\gamma = 2$ ballistic motion
      - $\gamma = 3$ turbulent diffusion
    - Hamiltonian systems: $1 < \gamma < 2$ (Hamiltonian intermittency)
  - Motion governed by Fokker-Planck-Kolmogorov diffusion equation

Lyapunov Exponent

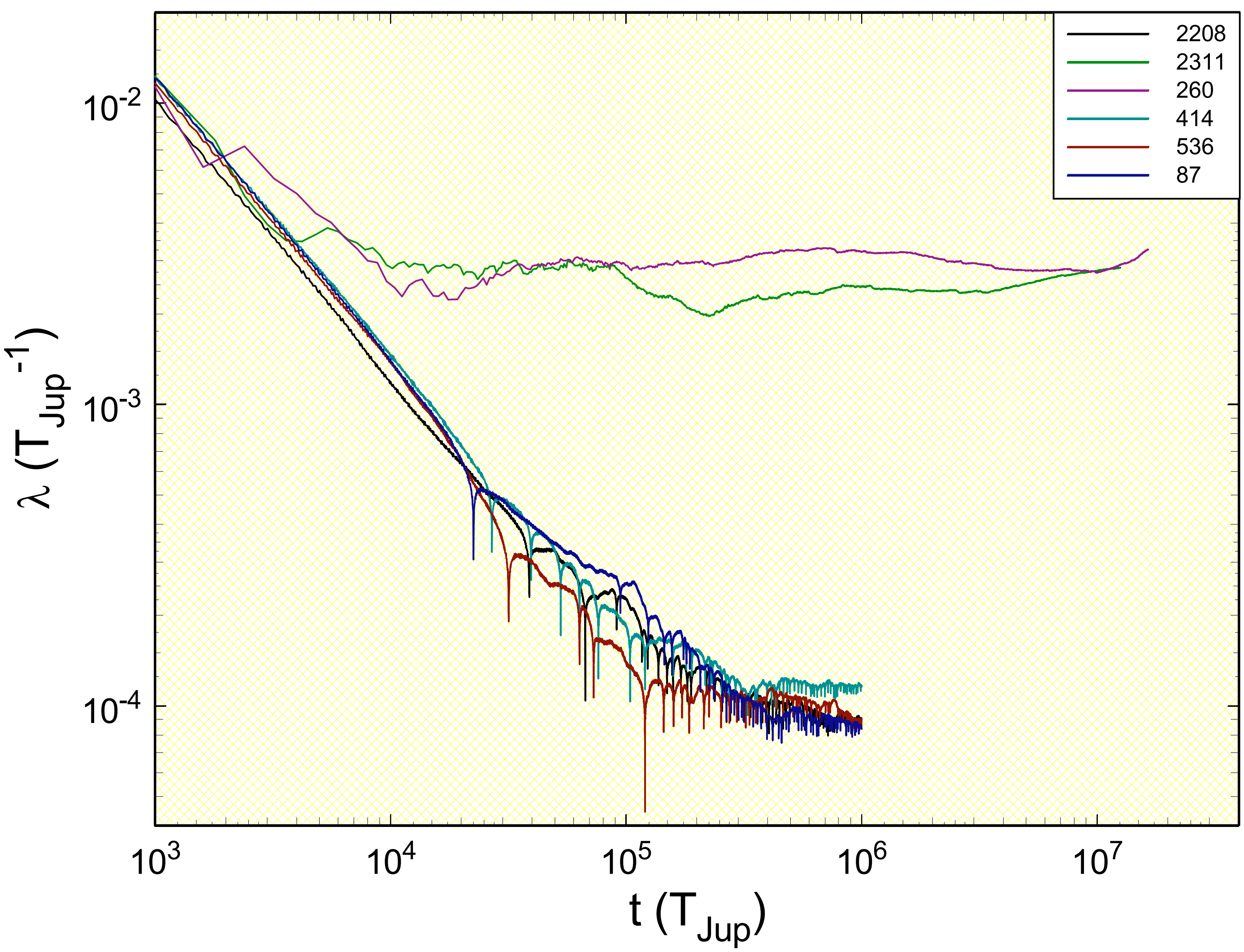
- Lyapunov exponent λ:

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \frac{d(t)}{d(0)}$$



Lyapunov Exponent

- Typical Behavior

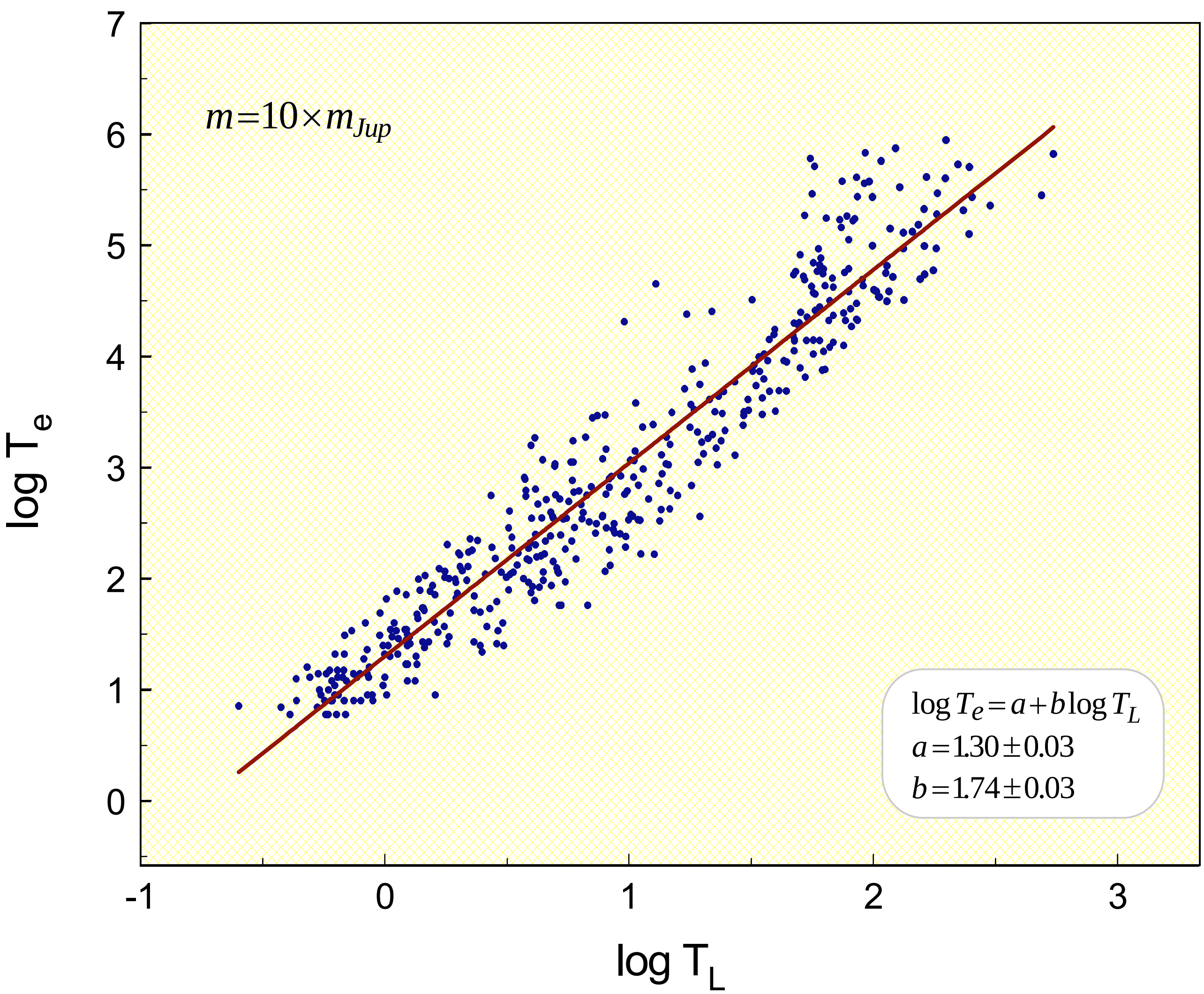


Lyapunov Exponent Relation

- T_L-T_e relation:

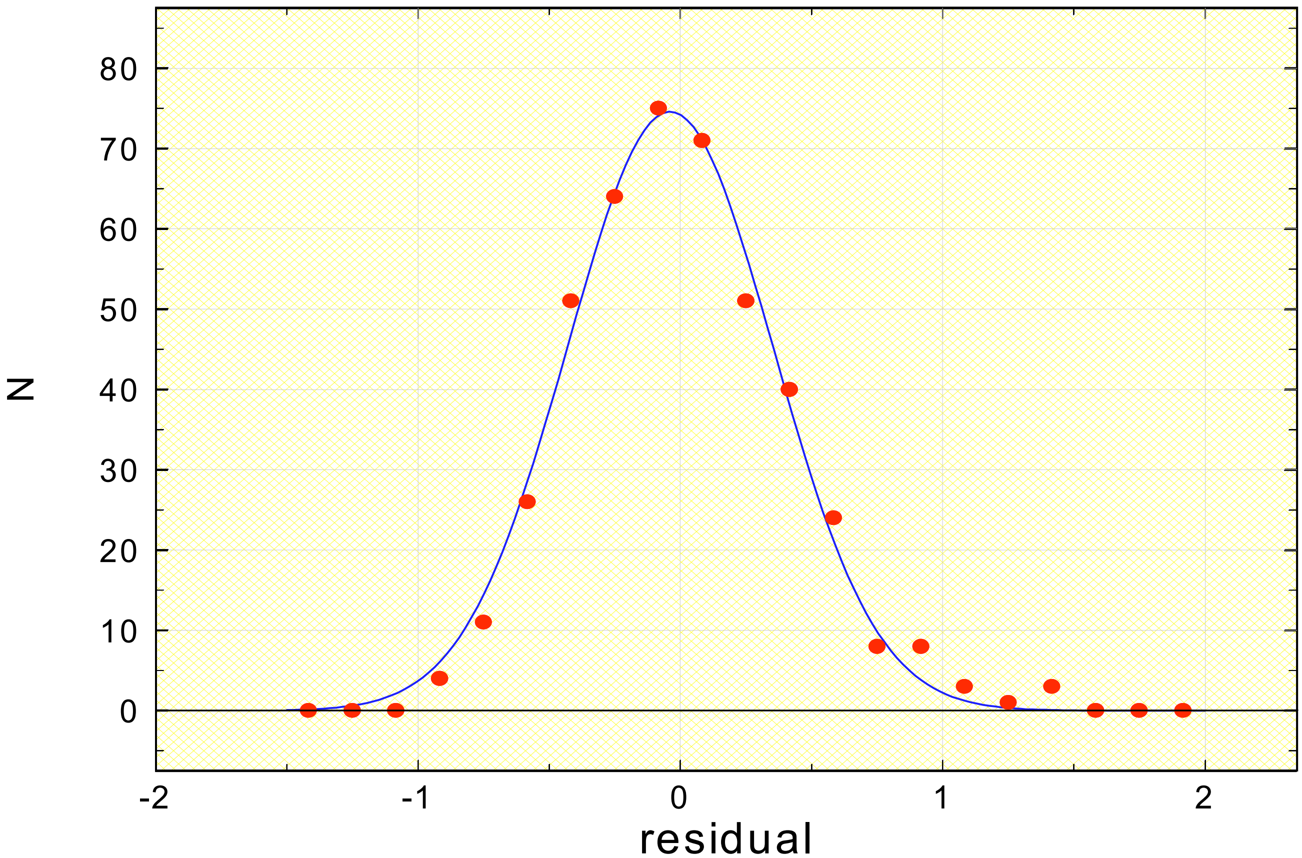
$$\log\frac{T_e}{T_0} = a + b\log\frac{T_L}{T_0}$$

- Empirical



Lyapunov Exponent Relation

- Well-determined



Lyapunov Exponent Relation

- Getting λ takes *orders of magnitude* less CPU time
- Potential predictive uses:
  - disruptive stability of dynamical systems
  - Hilda group asteroids (Franklin et al. 1993)
    - T_L for real asteroids all consistent with lifetimes greater than T_{SS}
    - T_L of objects with libration amplitude above those observed imply $T_e < T_{SS}$
    - Objects in 3:2 resonance with small proper eccentricities (where no actual asteroids exist): T_L implies $T_e \lesssim T_{SS}$

Lyapunov Exponent Relation

- (Potential predictive uses)
  - outer-belt asteroids (Murison et al. 1994)
    - Isolated stragglers consistent with expected tail members of Gaussian distribution
      - Gaussian implies "normal" diffusion (FP equation)
    - Expected "event" times all within 3σ of T_{SS}
    - Should see eventual depletion due to evaporation of tail members

Problems with the Relation

- Normalizing timescale T_0
  - rescales a
- Still can need long integrations: 3σ in a power law exponent

$$\frac{T_e}{T_0} = A \cdot \left(\frac{T_L}{T_0} \right)^b$$

where

$$\log A = a$$

- Calibration
- **When is the relation applicable?**

Applicability

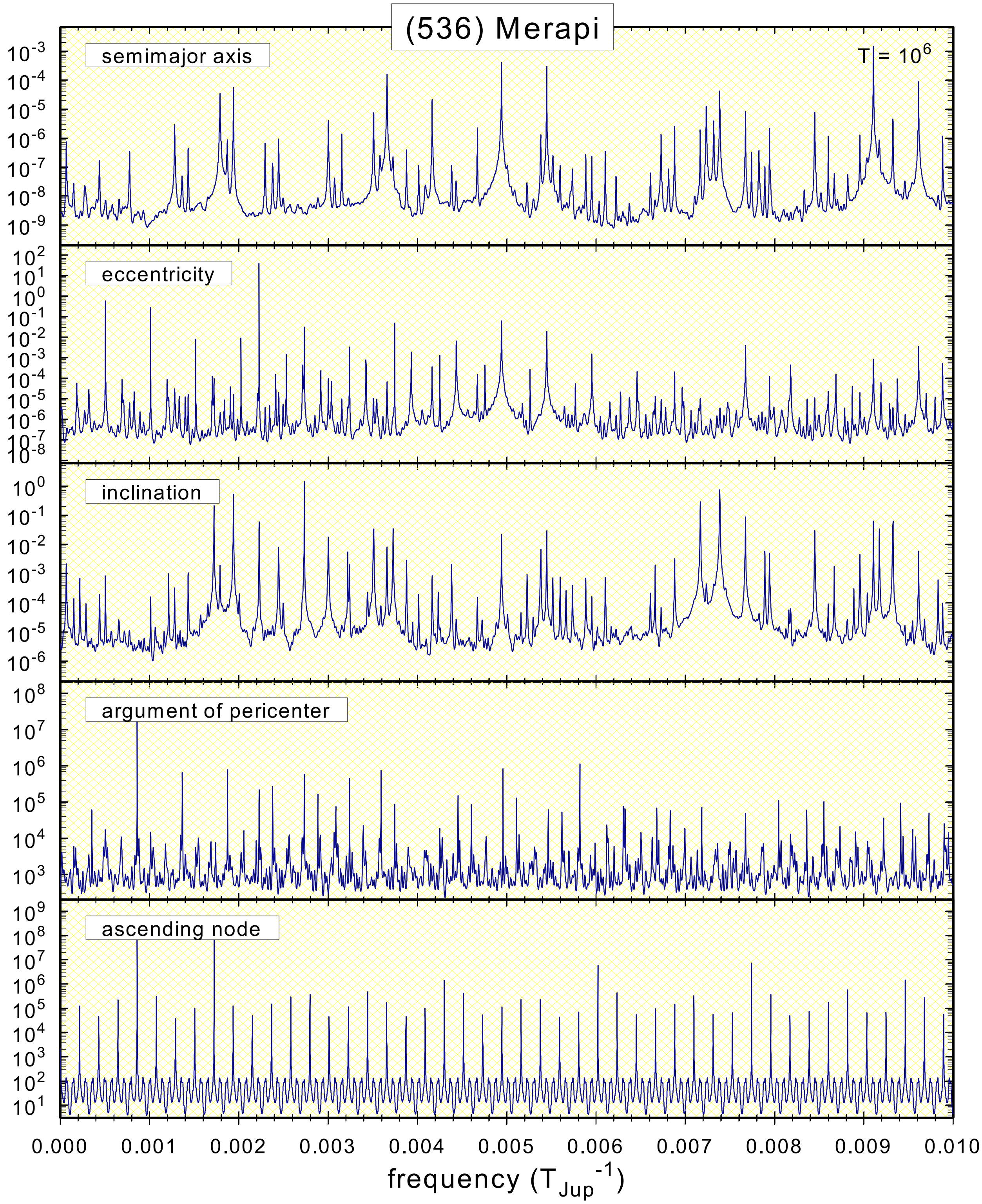
- Holds in strong overlap regime
- Unreliable or invalid in isolated resonance regime?

New Results – Severe Chaos

- Stable for long periods
  - integrated (260) Huberta for 500 Myr
  - integrated (2311) El Leoncito for 210 Myr
  - "3σ" outer belt asteroids?
- Examples easy to find in outer belt
  - 21 of 25 with T_L < 60,000 yr
- Characteristics:
  - Very large λ
  - Very broad power spectral features
  - Confinement near hypersurfaces

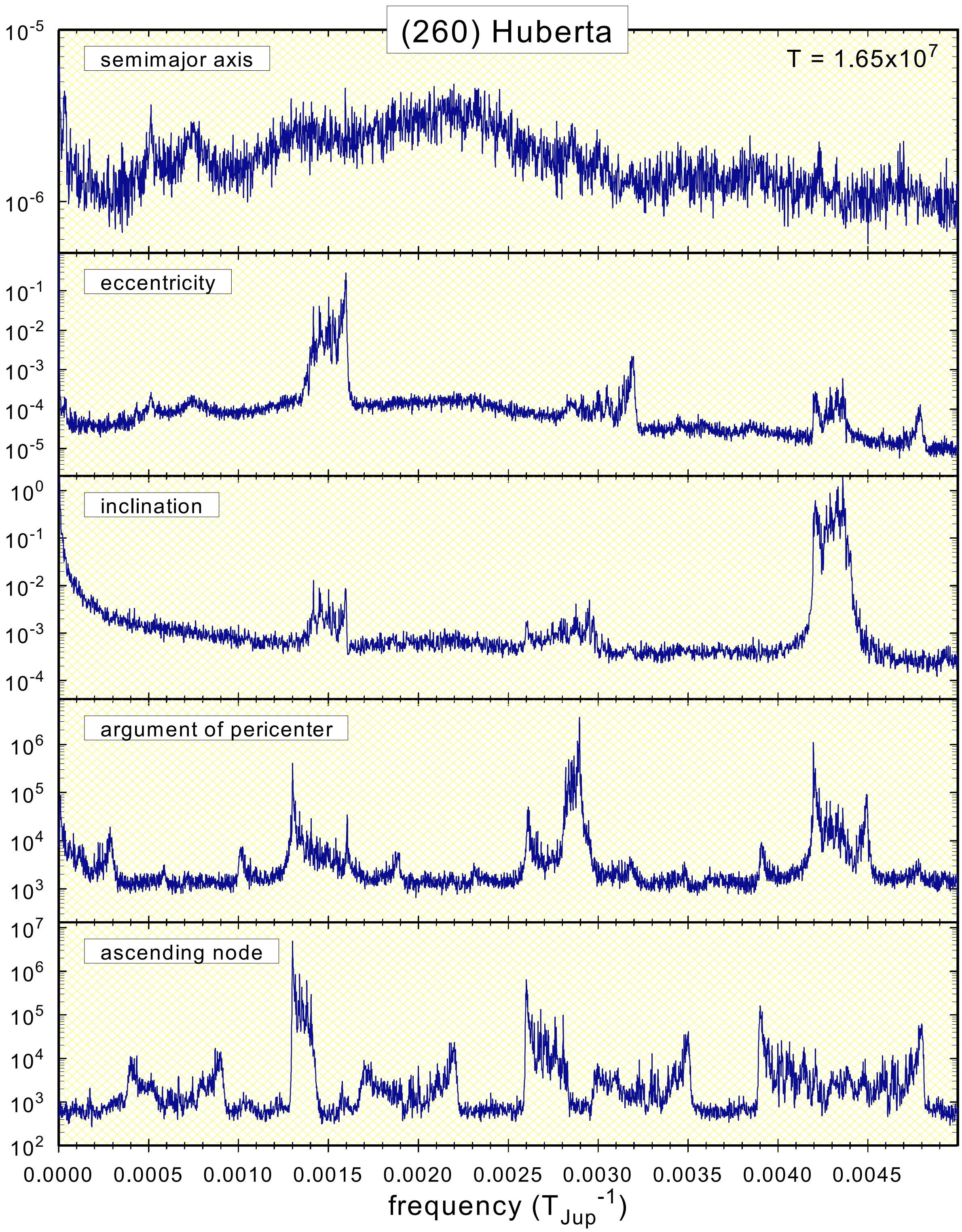
New Results – Severe Chaos

- Broad power spectral features



New Results – Severe Chaos

- *Extremely* broad power spectral features



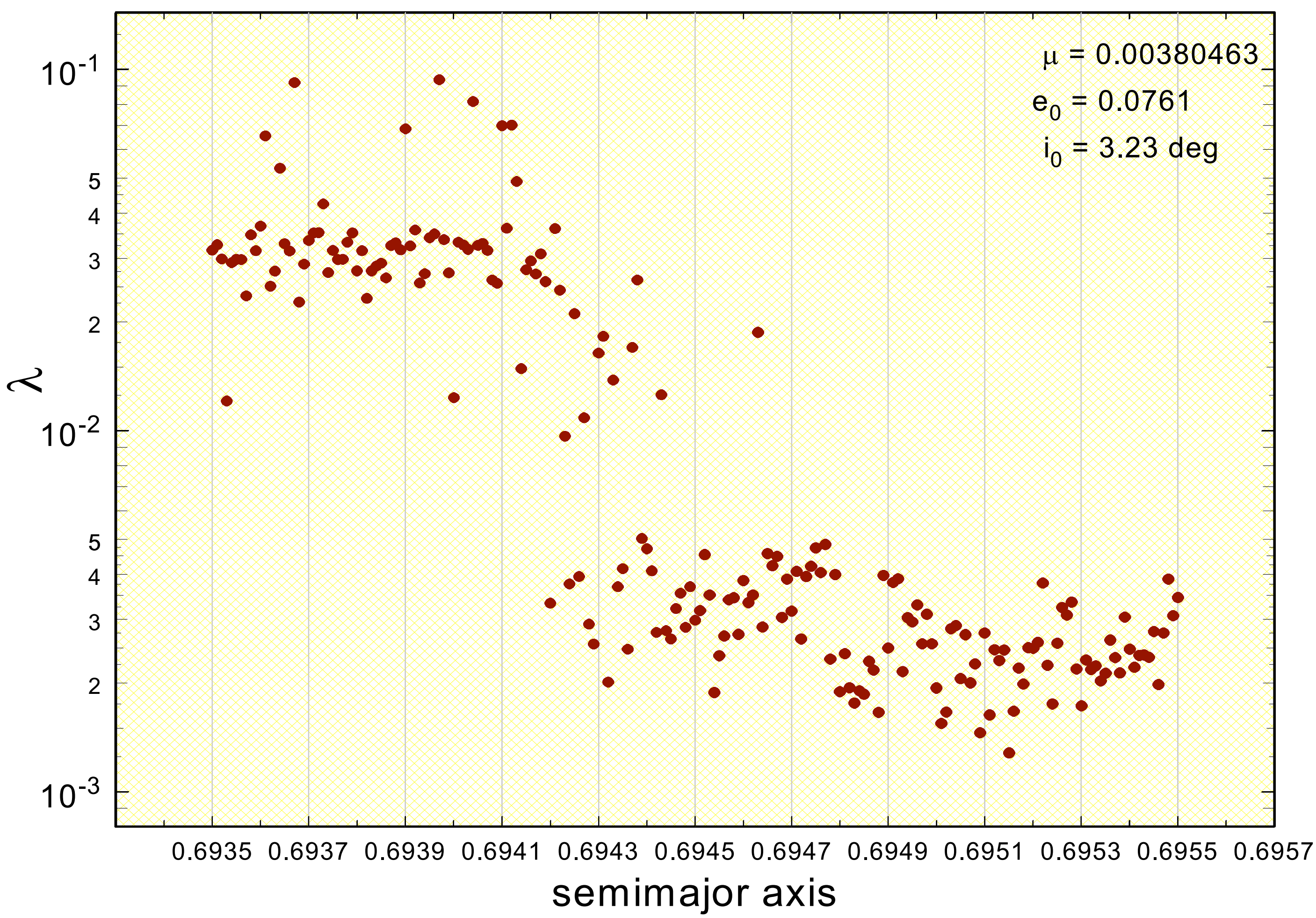
New Results – Severe Chaos

- Confinement near hypersurfaces

New Results – Overlap Threshold

- Previous results were in the overlap regime!
- Transition is abrupt
- Agreement with Tremaine's approximation

$$\Delta a \approx 1.49\,\mu^{2/7}$$



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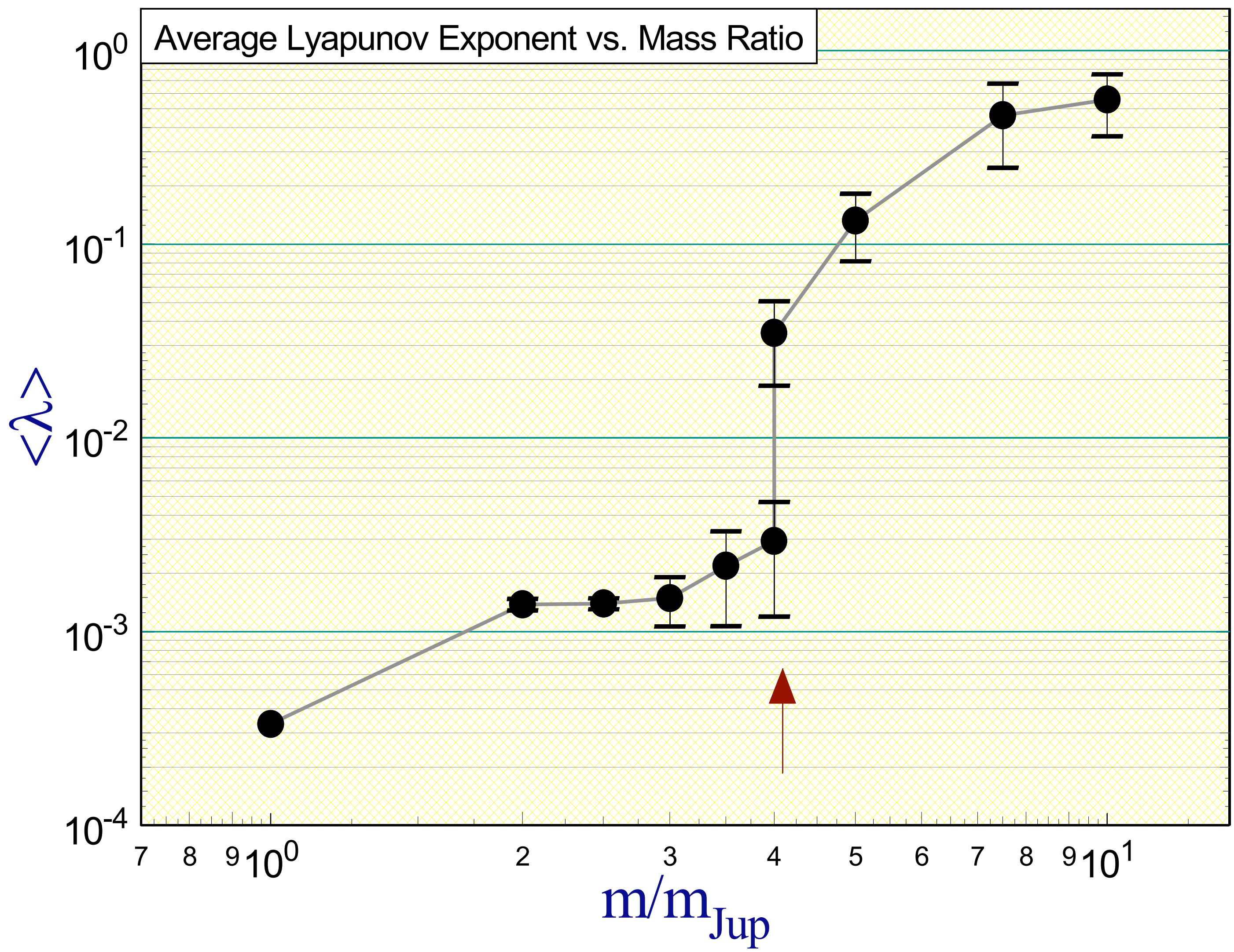
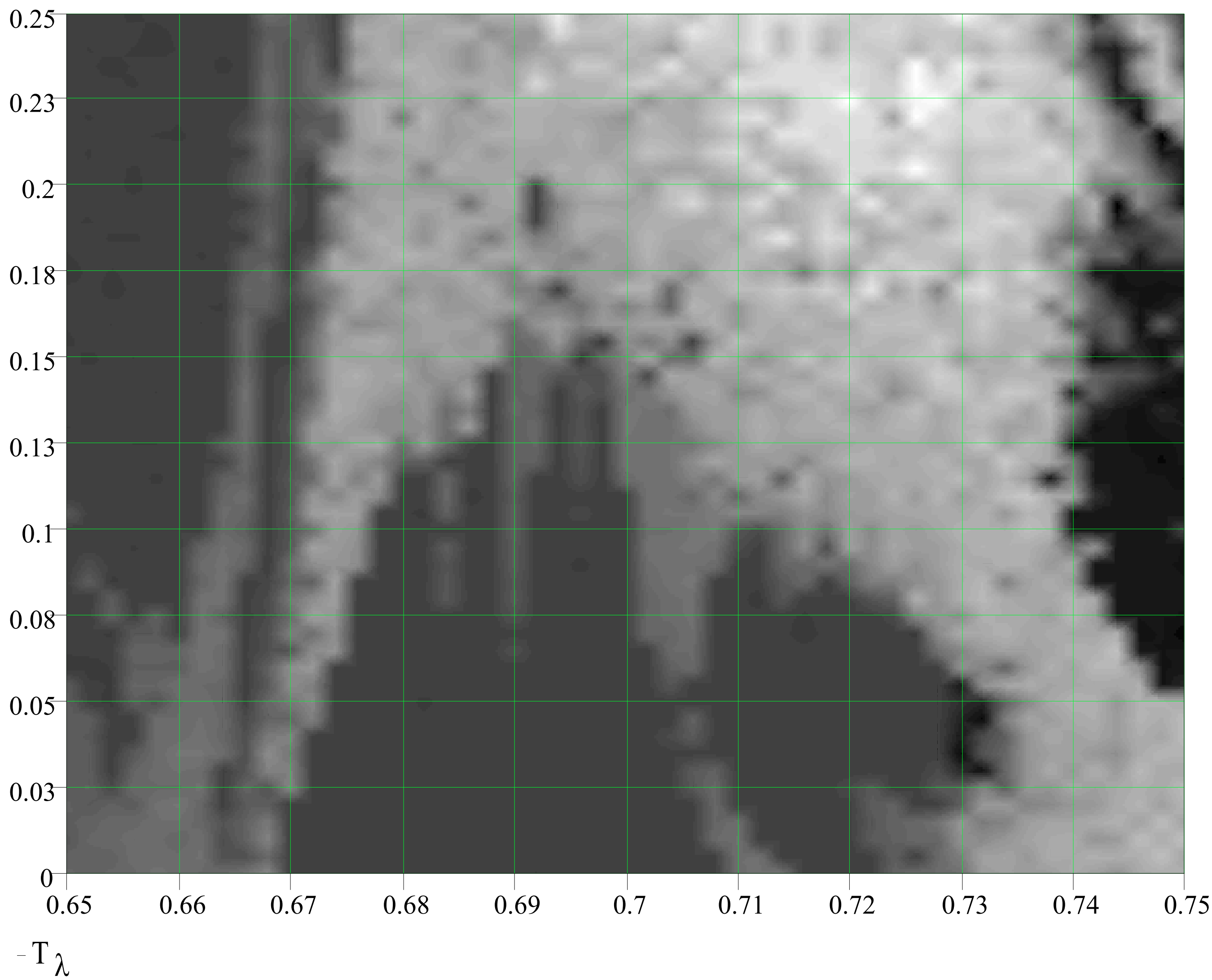


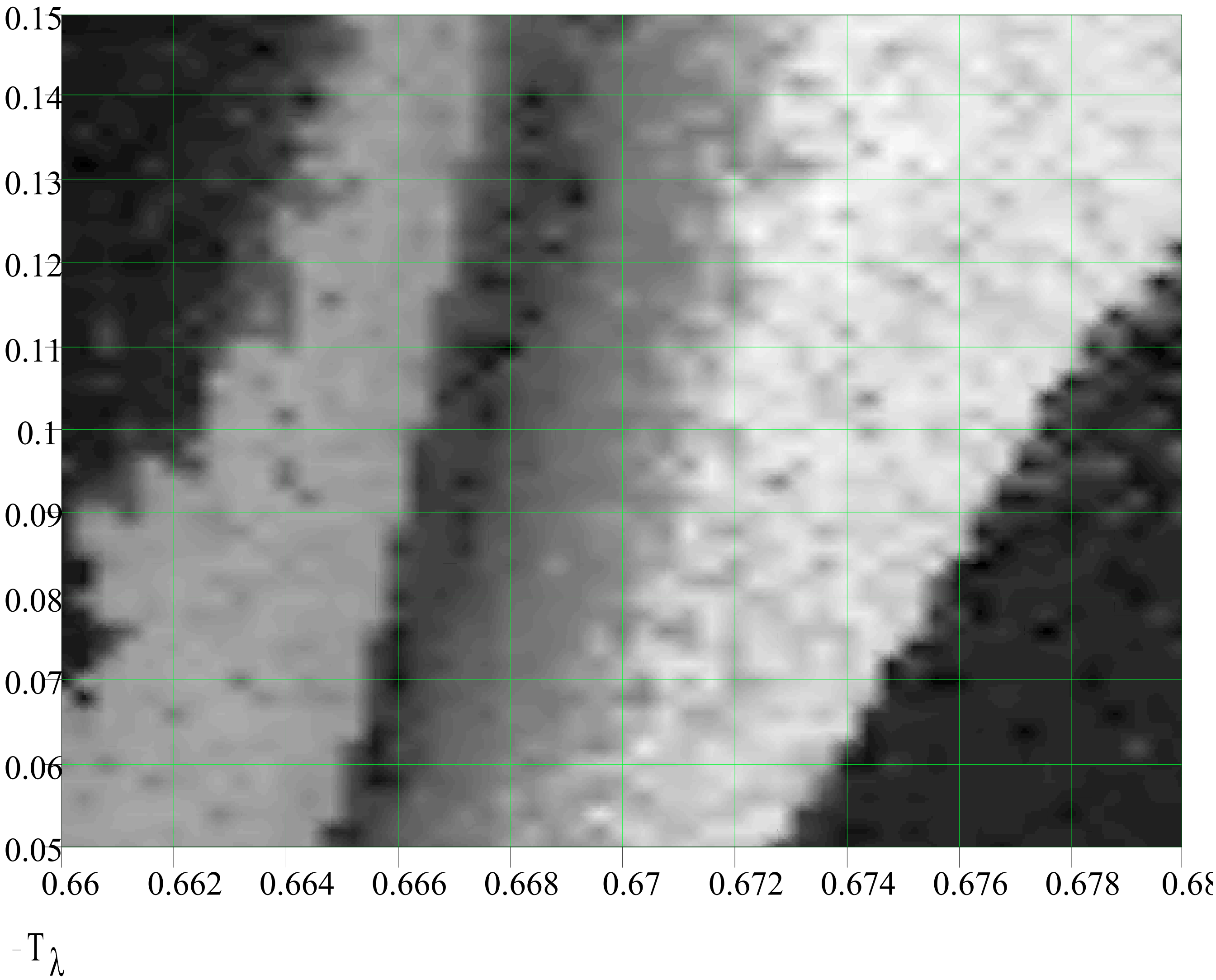
New Results – Overlap Threshold

- Evidence that relation is valid in overlap regime but more complicated in isolated resonance regime:
  - abrupt change in λ across threshold
  - possible (nearly?) bounded chaos examples (cf. confinement to hypersurfaces)
  - Varvoglis, Konishi, and FP diffusion

New Results – Stable Regions



New Results – Stable Regions

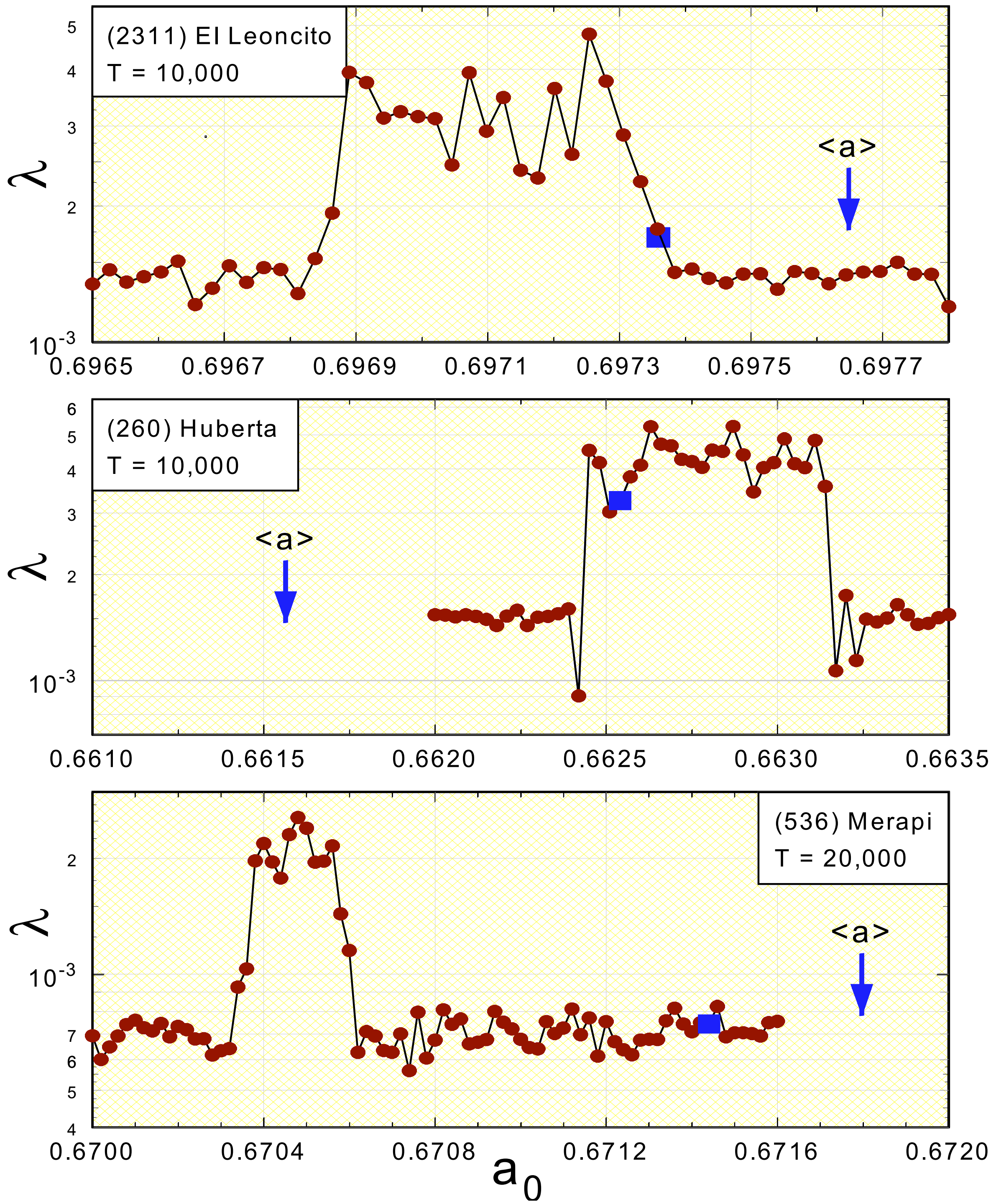


New Results – Stable Regions

- Structures are smooth functions of perturbing strength (mass ratio)
- Structures formed by families of periodic orbits (characteristic curves)
- Fractal or fractal-like
- <a> and <b> from Lyapunov relation are functions of
  - e_p
  - μ (perturbing strength)
  - (a_0,e_0)

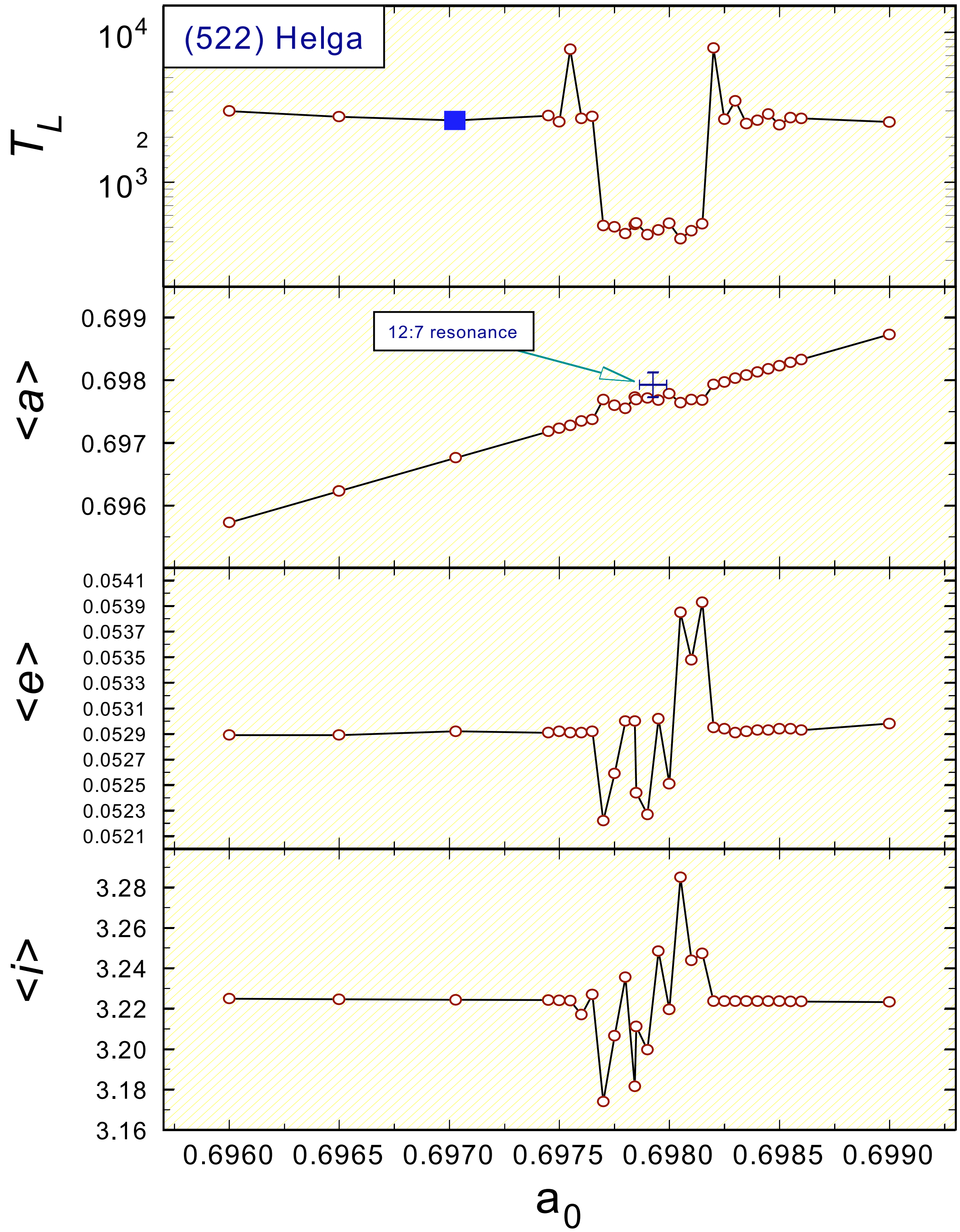
New Results – Resonances

- Large values of λ associated with high order mean motion resonances



New Results – Resonances

- (Correlation between λ and resonances)



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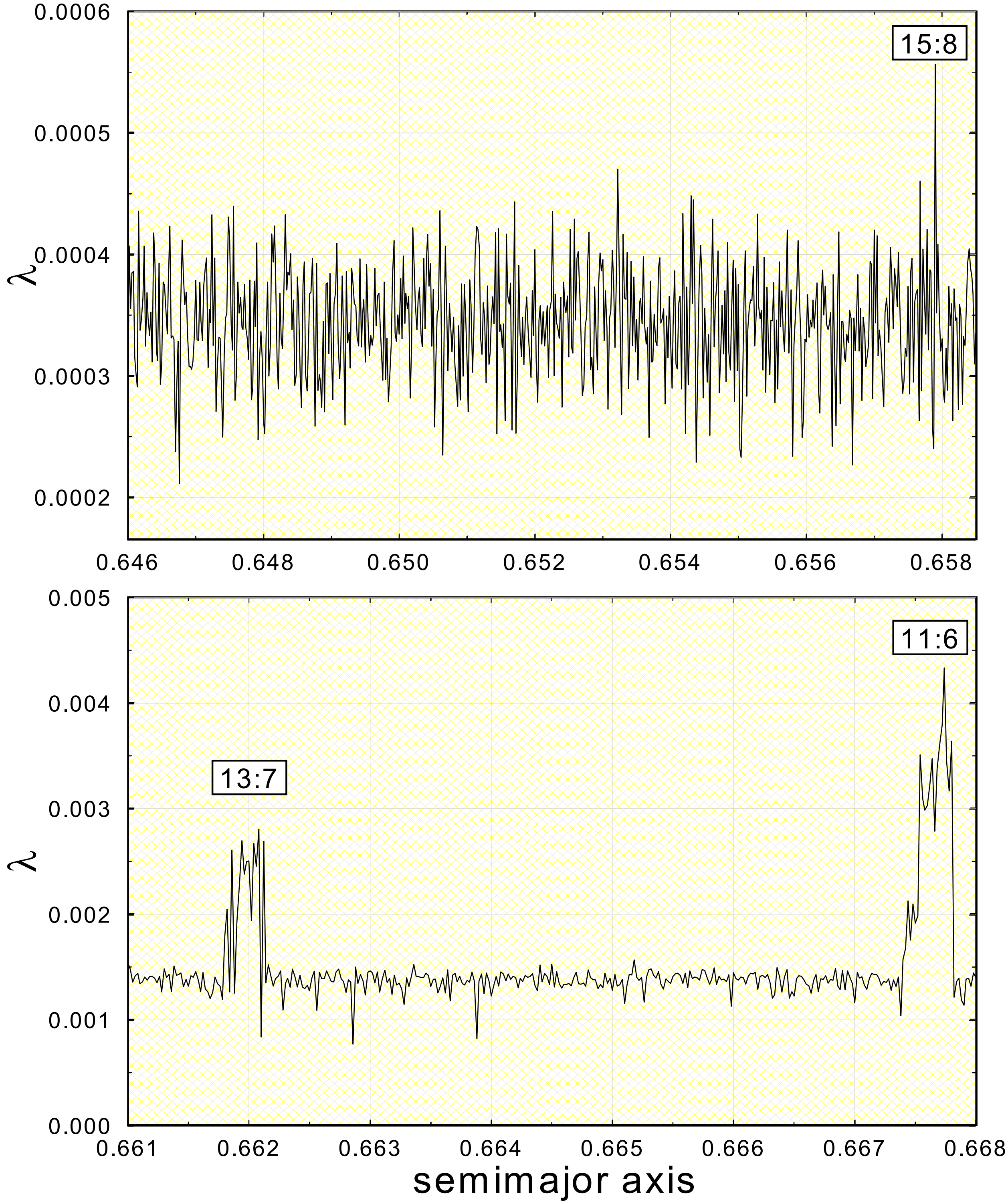
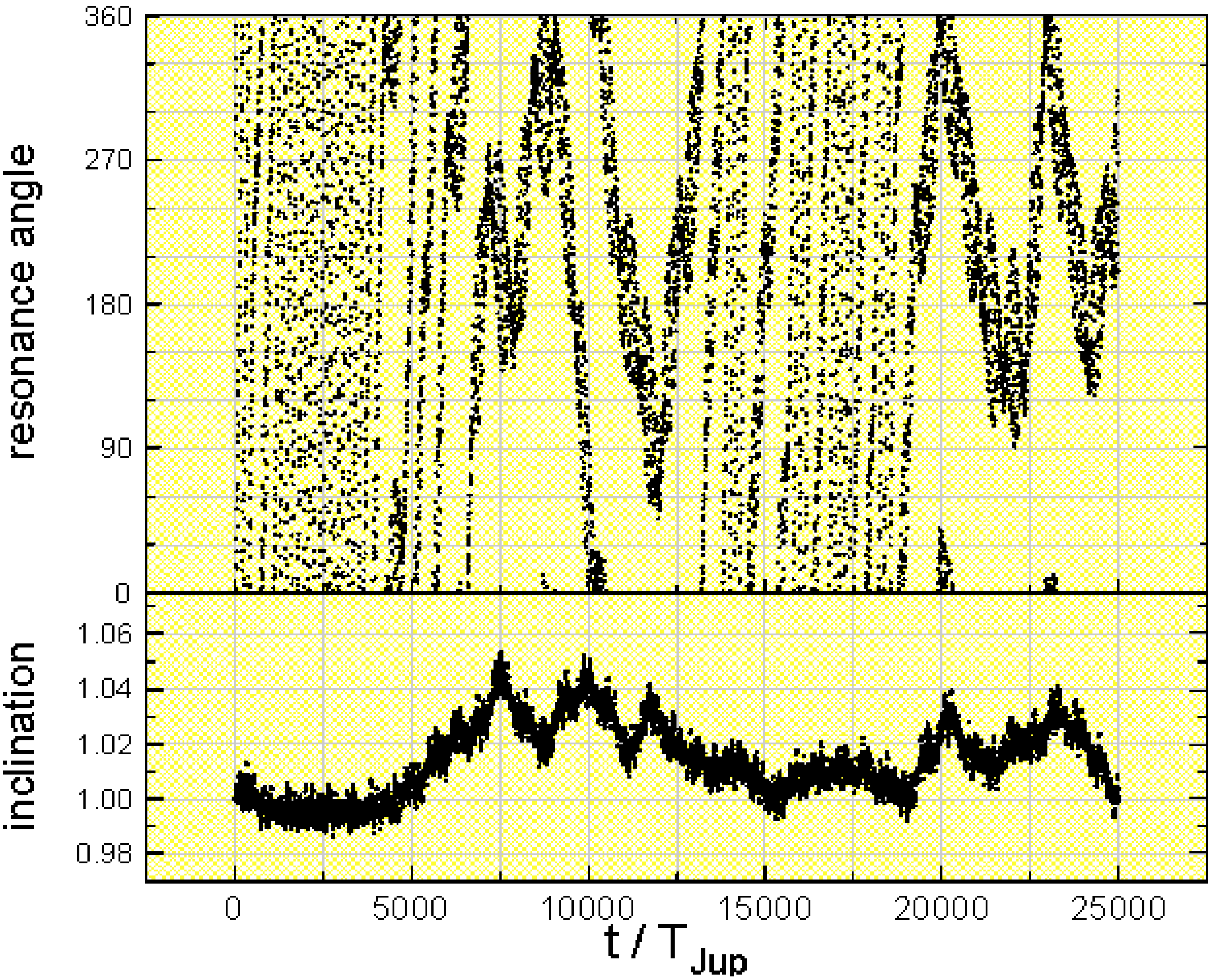


New Results – Resonances

- Resonance angle

$$\sigma = (p+q)\cdot \nu_{Jup} - p\cdot(\nu+\omega+\Omega)$$



Summary

- Lyapunov exponent relation probably applicable only in regions where phase space barriers have been destroyed by perturbations
  - Resonance overlap regime
  - "Global" sea of chaos
- Overlapping resonance transition is abrupt
- Orbits with very large λ are common in isolated (high-order) resonance regime
- Chaotic motion is confined between porous nested hypersurfaces in the isolated resonance regime
- Objects in high-order mean-motion resonances exhibit stronger chaos